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THE MOTION OF A MULTILINK SYSTEM ALONG A HORIZONTAL PLANE†

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The possible motions of a plane multilink system along a horizontal plane are investigated. Forces of dry friction act between the multilink system and the plane, obeying Coulomb's law. The multilink system is driven by internal control torques acting at the joints connecting the links. Control methods are constructed that ensure motion of the multilink system as a whole in any given direction. The velocity of these motions is estimated. The forms of motion found can serve as a model of the movement of snakes and other animals, and can also be used in designing snake-like robotic systems.

1. INTRODUCTION

We will consider different ways in which bodies (apparatus, vehicles and animals) can move along a plane horizontal surface. Suppose there are no external forces, apart from gravitational and surface reaction forces, and also no reactive forces. Then, as is well known, friction forces at the points of contact of the body with the surface play an important role: it is only when these forces are present that motion can begin at all. A distinguishing feature of most known methods of motion, using wheels, legs or caterpillar tracks, is as follows: during the motion, different points of the body come into contact with the surface. In fact, during the rolling of a wheel the contact points change continuously, during walking they change from step to step, and so on. In those rare cases where the contact points do not change (for example, it is possible to ski without removing the skis from the snow), redistribution of the normal pressure forces occurs between the points of contact (the skier transfers his weight from one leg to the other).

However, there is a method of motion where the contact points remain unchanged and, furthermore, where the normal pressure of the body on the surface at these points also does not change. This method of motion occurs in snakes and some other animals. They are in constant contact with the surface over their entire (or almost entire) length and move merely by bending their body. These animals can move both forwards and sideways, i.e. perpendicular to the axis of their body. Different aspects of the mechanics of snakes, and also certain problems of the mechanics of robots using this principle of motion, were discussed in [1-3].

A distinguishing feature of this method of motion is that the control torques driving the body are applied to axes perpendicular to the plane of motion. In other words, all motion of the system is planar.

The present paper proposes a mechanical model of this method of motion. For the case of a simple three-link system it is shown that, purely by means of internal torques acting at the joints, it is possible to ensure that the system moves in any given direction or rotates on the spot. It is assumed that dry friction forces act between the multilink system and the plane. The periodic laws of variation of the joint angles and the corresponding forms of motion of the multilink system are constructed and the displacements and velocities of motion are estimated.

2. THE MECHANICAL MODEL

Consider a plane three-link system $O_1C_1C_2O_2$ moving along a horizontal plane (Fig. 1). For simplicity, we will assume that the entire mass of the system is concentrated at the points O_1, C_1, C_2 and O_2 which are sliding along the plane. It is assumed that the links are absolutely rigid rods, and that their mass is negligible. The masses concentrated at the joints C_1 and C_2 will be denoted by m_1, and the mass of each of the end points O_1 and O_2 will be denoted by m_0. Thus, the mass of the entire multilink system is equal to $m = 2(m_0 + m_1)$. The lengths of the links O_1C_1 and C_2O_2 are assumed to be equal and are denoted by l, and the length of the link C_1C_2 is denoted by $2a$. The link C_1C_2, together with the masses

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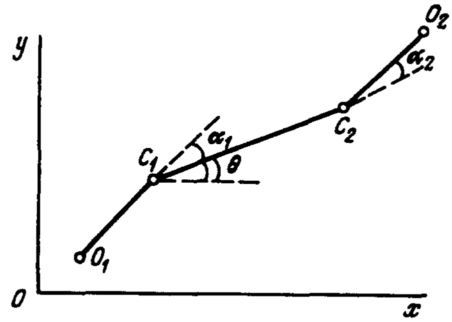

Fig. 1.

concentrated at the joints C_1 and C_2, will be referred to as the body, and the links O_1C_1 and C_2O_2, together with the end masses, will be termed the end links.

In the plane of motion, we will introduce a fixed Cartesian system of coordinates Oxy. The Cartesian coordinates of the centre of mass of the body C_1C_2 will be denoted by x, y, and the angles of inclination of the links O_1C_1, C_1C_2 and C_2O_2 to the x axis will be denoted by θ_1, θ and θ_2 respectively. We then have

$$\theta_1 = \theta + \alpha_1, \quad \theta_2 = \theta + \alpha_2 \tag{2.1}$$

where α_1 and α_2 are the angles between the body and the end links O_1C_1 and C_2O_2 respectively (see Fig. 1). The Cartesian coordinates of the joints C_1 and C_2 and the end points O_1 and O_2 are represented as follows

$$\begin{aligned} &x_i^C = x \mp a\cos\theta, \quad y_i^C = y \mp a\sin\theta \\ &x_i^O = x \mp a\cos\theta \mp l\cos\theta_i, \quad y_i^O = y \mp a\sin\theta \mp l\sin\theta_i \end{aligned} \tag{2.2}$$

Here and below, the upper minus and plus signs correspond to $i = 1$, and the lower signs to $i = 2$. Using Eqs (2.2), we define the coordinates of the centre of mass C of the system

$$\begin{aligned} &x_C = x - m_0 m^{-1} l(\cos\theta_1 - \cos\theta_2) \\ &y_C = y - m_0 m^{-1} l(\sin\theta_1 - \sin\theta_2), \quad m = 2(m_0 + m_1) \end{aligned} \tag{2.3}$$

Differentiating relations (2.2), we find the components of the velocities of the points C_1, C_2, O_1 and O_2

$$\begin{aligned} &v_{xi}^C = \dot{x} \pm a\dot{\theta}\sin\theta, \quad v_{yi}^C = \dot{y} \mp a\dot{\theta}\cos\theta \\ &v_{xi}^O = \dot{x} \pm (a\dot{\theta}\sin\theta + l\dot{\theta}_i\sin\theta_i), \quad v_{yi}^O = \dot{y} \mp (a\dot{\theta}\cos\theta + l\dot{\theta}_i\cos\theta_i) \end{aligned} \tag{2.4}$$

The angular momentum of the multilink system about the origin of coordinates is calculated from the formula

$$K = \sum_{i=1}^{2} \left(m_0 \begin{vmatrix} x_i^O & y_i^O \\ v_{xi}^O & v_{yi}^O \end{vmatrix} + m_1 \begin{vmatrix} x_i^C & y_i^C \\ v_{xi}^C & v_{yi}^C \end{vmatrix} \right) \tag{2.5}$$

Substituting relations (2.2) and (2.4) into expression (2.5), after lengthy but elementary reduction we obtain

$$\begin{aligned} &K = m(x\dot{y} - y\dot{x}) + 2(m_0 l^2 + m_0 a^2 + m_1 a^2)\dot{\theta} + m_0 l^2(\dot{\alpha}_1 + \dot{\alpha}_2) + \\ &+ m_0 l[\dot{x}(\sin\theta_1 - \sin\theta_2) - \dot{y}(\cos\theta_1 - \cos\theta_2)] + m_0 a l\dot{\theta}(\cos\alpha_1 + \cos\alpha_2) + \\ &+ m_0 l\dot{\theta}_1(-x\cos\theta_1 - y\sin\theta_1 + a\cos\alpha_1) + m_0 l\dot{\theta}_2(x\cos\theta_2 + y\sin\theta_2 + a\cos\alpha_2) \end{aligned} \tag{2.6}$$

For each of the masses O_1, C_1, C_2 and O_2, its weight is counterbalanced by the normal reaction of the plane. The dry friction force acting on each of these masses obeys Coulomb's law

$$\mathbf{F}_i = -m_i gk\mathbf{v}_i v_i^{-1}, \quad v_i \neq 0; \quad |\mathbf{F}_i| \leq m_i gk, \quad v_i = 0 \tag{2.7}$$

where $\mathbf{F}_i$ is the two-dimensional friction force vector, lying in the Oxy plane, $\mathbf{v}_i$ is the velocity vector of the point considered, m_i is its mass, g is the acceleration due to gravity and k is the coefficient of friction between the masses and the plane.

At the joints C_1 and C_2, there are control torques M_1 and M_2, respectively, created by motors located at these joints. We will assume that the moments M_1 and M_2 are applied to the end links O_1C_1 and O_2C_2 respectively. Accordingly, the torques $(-M_1)$ and $(-M_2)$ are applied to the body by these links.

The purpose of the present paper is to construct control laws ensuring the motion of a multilink system along a plane in any direction. To do this, it is sufficient to construct the laws of motion of a multilink system with an initial rectilinear configuration along itself (longitudinal motion) and across itself (lateral motion), and also its rotation on the spot. Using these forms of motion, it is easy to implement any movement.

3. ELEMENTARY MOTIONS

The motions required will be constructed as combinations of simpler motions, which will be referred to as elementary motions. Elementary motions begin from a state of rest and end, likewise, in a state of rest. The initial and final values of the angle of rotation of a link, α_i, for each elementary motion will be denoted by α_i^0 and α_i^1 respectively, where $i = 1, 2$. We also introduce the notation

$$\Delta\alpha_i = \alpha_i^1 - \alpha_i^0, \quad i = 1, 2 \tag{3.1}$$

The elementary motions are divided into slow and rapid motions.

In the slow motions, one or both end links move, their angular velocities and accelerations being sufficiently small for the body C_1C_2 not to be involved in the motion. The angular velocity of rotation of each link, $\dot{\alpha}_i$ ($i = 1, 2$), retains a constant sign. To fix our ideas, we assume that the absolute value ω of the angular velocity of the links, $\dot{\alpha}_i$ ($i = 1, 2$), first increases from 0 to ω_0 and then decreases from ω_0 to 0. The absolute value of the angular acceleration $|\dot{\omega}|$ is assumed to be constant and equal to ε_0. The following relations hold

$$\begin{gathered} |\dot{\alpha}_i| = \omega, \quad |\Delta\alpha_i| = \omega_0 T/2, \quad i = 1, 2, \quad \omega_0 = \varepsilon_0 T/2 \\ \omega(t) = \varepsilon_0 t, \quad t \in [0, T/2]; \quad \omega(t) = \varepsilon_0 (T - t), \quad t \in [T/2, T] \end{gathered} \tag{3.2}$$

where T is the duration of the slow motion. The graph of $\omega(t)$ is shown in Fig. 2. Either one end link takes part in the slow motions, the other remaining fixed, or both links move synchronously. In the latter case, they can rotate either in the same direction or in opposite directions, in which case the following condition is satisfied

$$\alpha_2(t) = \pm\alpha_1(t) + \beta, \quad t \in [0, T] \tag{3.3}$$

where β is a constant. Both links begin and stop moving simultaneously, in the same time T, and relations (3.2) apply for each of them.

In the rapid motions, the angular velocities and accelerations of the end links are fairly high, and the time of motion is short compared with the time of the slow motions T. Here, the values of the control torques M_1 and M_2 at joints C_1 and C_2, respectively, should be high compared with the moments of the friction forces, which are limited to values of the order of μgkL, where $\mu = \max(m_0, m_1)$ and $L = \max(a, l)$. For this reason, when considering the rapid motions, one can neglect friction forces. During the rapid motions, both end links rotate either in the same direction or in opposite directions synchronously. Here, relations (3.3) are again satisfied, but additional conditions also apply.

We shall examine the following three types of rapid motion.

1. The end links rotate in opposite directions, and, at the beginning and end of the motion, one of the angles α_i ($i = 1, 2$) is zero. Taking account of condition (3.3), we have $\alpha_2(t) + \alpha_1(t) = \beta$ and

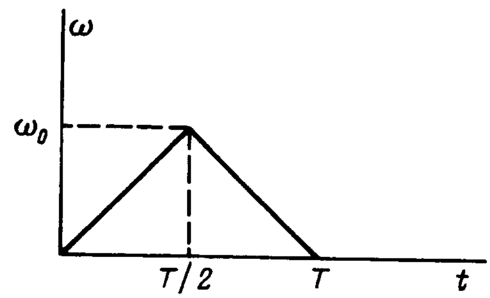

Fig. 2.

$$\alpha_1^0 = 0, \quad \alpha_2^0 = \beta \quad \text{or} \quad \alpha_1^0 = \beta, \quad \alpha_2^0 = 0 \tag{3.4}$$

2. The end links rotate in opposite directions, with $\beta = 0$, so that $\alpha_2(t) = -\alpha_1(t)$.
3. The end links rotate in the same direction, with $\beta = 0$, so that $\alpha_2(t) = -\alpha_1(t)$.

The variation of the angular velocities of the end links for the rapid motions is unimportant. What is important is simply the fact that the motions of the two links occur for the same time, satisfying the above conditions, and the fact that they begin and end in a state of rest.

4. ANALYSIS OF THE SLOW MOTIONS

Let us examine the balance of forces and torques in the slow motions and find the sufficient conditions for the body to remain fixed in these motions.

Forces N_1, R_1 and N_2, R_2, respectively, act from the rotating end links O_1C_1 and O_2C_2 on the fixed body C_1C_2. These forces (in all cases, $i = 1, 2$) are

$$N_i = m_0\dot{\alpha}_i^2 l, \quad R_i = m_0\ddot{\alpha}_i l \tag{4.1}$$

Forces N_i are directed along the corresponding links O_iC_i, while the forces R_i are perpendicular to these links (see Fig. 3). Furthermore, the control torques equal to $(-M_i)$ are applied to the body. To fix our ideas, we will assume that the body is parallel to the x axis. The projections onto the x and y axes of the friction forces applied to the points C_i will be denoted by X_i and Y_i.

We will set up the equations of equilibrium of the body under the action of the applied forces and torques. We will take as the three equilibrium equations the conditions that the sums of the moments of the applied forces about the joints C_1 and C_2 are zero and the conditions that the sum of the projections of all forces onto the x axis is zero. We then have

$$2a(N_i \sin\alpha_i - R_i \cos\alpha_i \mp Y_i) = M_1 + M_2 \tag{4.2}$$

$$N_1 \cos\alpha_1 + R_1 \sin\alpha_1 - N_2 \cos\alpha_2 - R_2 \sin\alpha_2 = X_1 + X_2 \tag{4.3}$$

Here and below, the convention assumed in Section 2 applies: the upper minus and plus signs correspond to $i = 1$, and the lower signs correspond to $i = 2$. The condition that the sum of the projections of all forces onto the y axis is zero is a consequence of Eqs (4.2).

Note that the system in question is statically indeterminate: for the four unknown reaction forces X_i, Y_i there are a total of only three equations (4.2), (4.3).

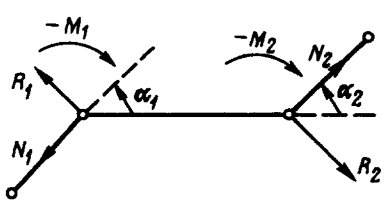

Fig. 3.

The projections onto the x and y axes of the overall forces Φ_i applied to the body by the end links O_iC_i will be denoted by Φ_{ix} and Φ_{iy}.

$$\Phi_{ix} = \mp(N_i \cos\alpha_i + R_i \sin\alpha_i), \quad \Phi_{iy} = \mp(N_i \sin\alpha_i - R_i \cos\alpha_i) \tag{4.4}$$

Using the notation (4.4), from (4.2) we find

$$Y_i = -\Phi_{iy} \mp Q, \quad Q = (M_1 + M_2)/(2a) \tag{4.5}$$

Equation (4.3) can be rewritten in the form

$$X_1 + X_2 = -\Phi_{1x} - \Phi_{2x} \tag{4.6}$$

Equilibrium of the body will occur if there are reactions X_i, Y_i satisfying relations (4.5), (4.6) and the inequalities of Coulomb's law (2.7)

$$(X_i^2 + Y_i^2)^{1/2} \leqslant F_1, \quad F_1 = m_1 gk \tag{4.7}$$

Thus, the forces Y_i are uniquely defined by relations (4.5), while the forces X_i satisfy the single equality (4.6). Furthermore, all of these forces must satisfy inequalities (4.7).

Since, to ensure equilibrium, it is sufficient to indicate one set of forces X_i, Y_i satisfying conditions (4.5)–(4.7), we assume that

$$X_i = -\Phi_{ix} \tag{4.8}$$

Equation (4.6) is satisfied in this case. Bearing in mind the derivation of the simple sufficient conditions of equilibrium, we will replace inequalities (4.7) with the more rigid conditions

$$|X_i| + |Y_i| \leqslant F_1 \tag{4.9}$$

which guarantee that inequalities (4.7) are satisfied. Substituting inequalities (4.5) and (4.8) into the left-hand side of inequality (4.9), we obtain the chain of inequalities

$$\begin{aligned} &|X_i| + |Y_i| \leqslant |\Phi_{ix}| + |\Phi_{iy}| + |Q| \leqslant \sqrt{2}(\Phi_{ix}^2 + \Phi_{iy}^2)^{1/2} + |Q| = \\ &= \sqrt{2}\,|\Phi_i| + |Q| = \sqrt{2}(N_i^2 + R_i^2)^{1/2} + |Q| = \\ &= \sqrt{2} m_0 l(\dot{\alpha}_i^4 + \ddot{\alpha}_i^2)^{1/2} + |Q| \leqslant \sqrt{2} m_0 l(\omega_0^4 + \varepsilon_0^2)^{1/2} + |Q| \end{aligned} \tag{4.10}$$

Here, use is made of the fact that the force Φ_i, with which end link O_iC_i acts on the body, is developed, on the one hand, into the components Φ_{ix}, Φ_{iy}, and, on the other hand, into the components N_i, R_i, determined by relations (4.1). Furthermore, estimates of the angular velocities and accelerations stemming from (3.2) are used.

We will now estimate the control torques M_i. We will write the equation of rotation of the end link O_iC_i in the form

$$m_0 l^2 \ddot{\alpha}_i = M_i - m_0 gkl \operatorname{sign}\dot{\alpha}_i \tag{4.11}$$

The following estimate results from (4.11) and (3.2)

$$|M_i(t)| \leqslant M_0, \quad M_0 = m_0 l(\varepsilon_0 l + gk) \tag{4.12}$$

We will estimate the value of Q from (4.5). If, in the slow motion, both end links rotate in the same direction, then $M_1 = M_2$ and from (4.5) and (4.12) we obtain

$$|Q| \leqslant M_0 a^{-1} \tag{4.13}$$

Estimate (4.13) obviously also holds when only one link rotates (the torque M_1 or M_2 is zero) and when the links rotate in opposite directions ($M_1 + M_2 = 0$). Substituting estimates (4.10), (4.12) and (4.13) into (4.9), we obtain the inequality

$$m_0 l[\sqrt{2}(\omega_0^4+\varepsilon_0^2)^{1/2}+(\varepsilon_0 l+gk)a^{-1}]\leqslant m_1 gk \tag{4.14}$$

which is the sufficient condition for the body to remain immobile for all slow motions.

A considerably better condition can be obtained for the slow motion during which the end links rotate in opposite directions. In this case we have $M_1 + M_2 = 0$ and $Q = 0$. Substituting (4.5) and (4.9) into the left-hand side of initial inequality (4.7), instead of (4.10) we have

$$(X_i^2+Y_i^2)^{1/2}=(\Phi_{ix}^2+\Phi_{iy}^2)^{1/2}=|\Phi_i|\leqslant m_0 l(\omega_0^4+\varepsilon_0^2)^{1/2} \tag{4.15}$$

Substituting inequality (4.15) into condition (4.7), we obtain the sufficient condition for the body to be immobile when the end links rotate in opposite directions

$$m_0 l(\omega_0^4+\varepsilon_0^2)^{1/2}\leqslant m_1 gk \tag{4.16}$$

Condition (4.16) is always satisfied for sufficiently slow rotations when ω_0 and ε_0 are sufficiently small. Condition (4.14) is obviously satisfied for fairly slow rotations if $m_0 l < m_1 a$.

5. ANALYSIS OF THE RAPID MOTIONS

As noted above, the rapid motions are implemented by large control torques, compared with which the moments of the friction forces are negligible. Therefore, for the rapid motions the influence of friction forces can be neglected. Since these motions begin from a state of rest, during the rapid motions the position of the centre of mass of the multilink system remains unchanged, and its angular momentum remains equal to zero. We then have

$$\dot{x}_C=0,\quad \dot{y}_C=0,\quad K=0 \tag{5.1}$$

Substituting relations (2.3) into the first two equalities of (5.1), we obtain

$$\begin{aligned}\dot{x}&=-m_0 m^{-1} l(\dot\theta_1\sin\theta_1-\dot\theta_2\sin\theta_2)\\ \dot{y}&=m_0 m^{-1} l(\dot\theta_1\cos\theta_1-\dot\theta_2\cos\theta_2)\end{aligned} \tag{5.2}$$

We transform expression (2.6) by substituting into it relations (5.2) and the equalities

$$\dot\theta_i=\dot\theta+\dot\alpha_i,\quad i=1,2,\quad \dot\alpha_2=\pm\dot\alpha_1 \tag{5.3}$$

which follow from (2.1) and conditions (3.3).

For the case where the end links rotate in opposite directions, selecting the minus sign in (5.3), we obtain

$$\begin{aligned}K&=2\{(m_0 l^2+m_0 a^2+m_1 a^2)-m_0^2 m^{-1} l^2[1-\cos(\alpha_1-\alpha_2)]+\\ &+m_0 al(\cos\alpha_1+\cos\alpha_2)\}\dot\theta+m_0 al(\dot\alpha_1\cos\alpha_1+\dot\alpha_2\cos\alpha_2)\end{aligned} \tag{5.4}$$

We will examine the motions of the body separately for rapid motions 1 and 2 from Section 3.

For motion 1, in accordance with condition (3.3), we make the change of variables

$$\alpha_1=\beta/2+\eta,\quad \alpha_2=\beta/2-\eta \tag{5.5}$$

The new variable η, according to conditions (3.3) and (3.4), changes during the motion either from $-\beta/2$ to $\beta/2$ or from $\beta/2$ to $-\beta/2$, i.e. always in the range $|\eta|\leqslant|\beta|/2$. After substituting (5.5), relation (5.4) takes the form

$$\begin{aligned}&K=2f(\eta)\dot\theta-2h(\eta)\dot\eta\\ &f(\eta)=m_0 l^2+m_0 a^2+m_1 a^2-2m_0^2 m^{-1} l^2\sin^2\eta+2m_0 al\cos(\beta/2)\cos\eta\\ &h(\eta)=m_0 al\sin(\beta/2)\sin\eta\end{aligned} \tag{5.6}$$

Since $K = 0$, by (5.1), from (5.6) we have a quadrature for the increment in the angle θ during the motion

$$\Delta\theta = \pm \int_{-\beta/2}^{\beta/2} \frac{h(\eta)}{f(\eta)} d\eta \tag{5.7}$$

The $\pm$ signs in (5.7) correspond to the two possible cases in (3.4). Since $f(\eta)$ is an even function and $h(\eta)$ is an odd function [see (5.6)], in both cases we have $\Delta\theta = 0$.

For motion 2, in relations (5.5) and (5.6) it must be assumed that $\beta = 0$. In this case we have $h(\eta) \equiv 0$, and the condition $K = 0$ leads to the relation $\dot{\theta} = 0$, $\theta = \text{const}$.

Thus, in motion 2, the angle of inclination of θ of the body to the x axis remains constant, and in motion 1 this angle oscillates, but its final value is the same as the initial value. We will select the orientation of the axes so that the x axis is parallel to the body at the beginning of motions 1 and 2. Then, for both of these motions we have $\theta = 0$ at the beginning and at the end of them.

We will determine the increments Δx and Δy of the coordinates of the middle of the body during the time of motions 1 and 2. Since, by (5.1), the displacement of the centre of mass for these motions is zero ($\Delta x_C = \Delta y_C = 0$), from (2.3) we have

$$\begin{aligned} \Delta x &= m_0 m^{-1} l \Delta(\cos\theta_1 - \cos\theta_2) \\ \Delta y &= m_0 m^{-1} l \Delta(\sin\theta_1 - \sin\theta_2) \end{aligned} \tag{5.8}$$

The symbol Δ in all cases denotes the total increment of the corresponding quantity during the motion.

We substitute relations (2.1) and (5.5) into (5.8) and bear in mind that $\theta = 0$ at the beginning and end of the motion. We obtain

$$\begin{aligned} \Delta x &= -2m_0 m^{-1} l \sin(\beta/2) \Delta \sin\eta \\ \Delta y &= 2m_0 m^{-1} l \cos(\beta/2) \Delta \sin\eta \end{aligned} \tag{5.9}$$

In motion 1, the quantity η changes either from $-\beta/2$ to $\beta/2$ or from $\beta/2 = -\beta/2$. Therefore, for motion 1, from (5.9) we obtain

$$\Delta x = \mp 4 m_0 m^{-1} l \sin^2(\beta/2), \quad \Delta y = \pm 2 m_0 m^{-1} l \sin\beta \tag{5.10}$$

where the choice of the upper and lower minus and plus signs corresponds to the two cases of (3.4).

In motion 2 we have $\beta = 0$ and $\eta = \alpha_1 = -\alpha_2$. From (5.9) we obtain

$$\Delta x = 0, \quad \Delta y = 2 m_0 m^{-1} l (\sin\alpha_1^1 - \sin\alpha_1^0) \tag{5.11}$$

It remains to examine rapid motion 3, in which $\alpha_2 = \alpha_1$ and, by (2.1), $\theta_2 = \theta_1$. From relations (5.2) we obtain $\dot{x} = \dot{y} = 0$, i.e., the middle of the body remains fixed. Having placed the origin of coordinates of the Oxy system in the middle of the body, we shall have $x = y = 0$. Expression (2.6) for the angular momentum in the case of motion 3 takes the form

$$K = 2(m_0 l^2 + m_0 a^2 + m_1 a^2 + 2 m_0 a l \cos\alpha_1)\dot{\theta} + 2 m_0 l (l + a\cos\alpha_1)\dot{\alpha}_1 \tag{5.12}$$

Since $K = 0$, from (5.12) we can define the increment $\Delta\theta$ in the form of a quadrature

$$\Delta\theta = -\int_{\alpha^0}^{\alpha^1} \varphi(\alpha_1) d\alpha_1 \tag{5.13}$$

The function $\varphi(\alpha_1)$ is equal to the ratio of the coefficients of $\dot{\alpha}_1$ and $\dot{\theta}$ in (5.12), while α^0 and α^1 are the initial and final values of the angles $\alpha_1 = \alpha_2$.

We will now form the longitudinal, lateral and rotational motions of the multilink system from the elementary motions described in Section 3.

6. LONGITUDINAL MOTION

At the initial instant of time, let the multilink system be at rest, and let all its links be parallel to the x axis. In this state we have $\theta = \alpha_1 = \alpha_2 = 0$ (see state 0 in Fig. 4).

The longitudinal motion can be provided by executing consecutively the following elementary motions.

1. Slow motion during which the link O_1C_1 rotates about the joint C_1 by an angle of γ. The other sections remain fixed, and the multilink system passes into state 1 in Fig. 4, in which $\alpha_1 = \gamma$ and $\alpha_2 = 0$.
2. Rapid motion of type 1 from Section 3, as a result of which the angle α_1 changes from γ to 0, and the angle α_2 changes from 0 to γ. The multilink system passes into state 2 in Fig. 4.
3. Slow motion, during which the angle α_1 changes from 0 to $-\gamma$, and the angle α_2 changes from γ to 0. The multilink system passes into state 3 in Fig. 4.
4. Rapid motion of type 1 from Section 3, as a result of which the angle α_1 changes from $-\gamma$ to 0, and the angle α_2 changes from 0 to $-\gamma$. The multilink system passes into state 4 in Fig. 4.
5. Slow motion, as a result of which the angle α_1 changes from 0 to γ, and the angle α_2 changes from $-\gamma$ to 0. The multilink system passes into state 5 in Fig. 4.

It is not difficult to see that state 5, in which $\alpha_1 = \gamma$ and $\alpha_2 = 0$, is identical with state 1 (see Fig. 4). Further, the cycle of two rapid and two slow motions, transferring the multilink system from state 1 to state 5, can be repeated any number of times. For the multilink system to be transferred from state 5 to the initial rectilinear state 0 at the end of the motion, it is necessary to perform the slow motion changing in the angle α_1 from γ to 0.

It will be recalled that each elementary motion begins and ends in a state of rest. We will determine the change in the position and orientation of the body as a result of one cycle of motion, transferring the multilink system from state 1 to state 5. Note that, in the course of slow motions, the body remains fixed, but the centre of mass of the multilink system moves along the x axis. This can be clearly seen from a comparison of the pairs of states 2, 3 and 4, 5 in Fig. 4: each time, the ends of the multilink system are displaced to the right along the x axis. In the course of rapid motions, on the other hand, the centre of mass of the multilink system remains fixed, but its body moves. As shown in Section 5, the orientation of the body does not change in the case of rapid motions of type 1 ($\Delta\theta = 0$), but the centre of the body undergoes displacements defined by formulae (5.10). For both rapid motions of type 1 that occur within the cycle, the second case of (3.4) occurs, so that the lower signs must be chosen in the formulae for the displacement (5.10). Here, the angle β from (5.10) is equal to $\pm\gamma$ for these two motions. Therefore, the complete change in y over the entire cycle is equal to zero, and the complete change $\Delta_0 x$ in the x coordinate is equal to

$$\Delta_0 x = 8m_0 m^{-1} l \sin^2(\gamma/2), \quad m = 2(m_0 + m_1) \tag{6.1}$$

Formula (6.1) defines the displacement of the multilink system over a cycle of the longitudinal motion. Since the duration of the rapid phases of motion is much shorter than the duration of the slow phases, the complete time of a cycle is approximately $2T$, where T is the duration of the slow motion. In the notation of (3.1) and (3.2) we have

$$|\Delta\alpha| = \gamma, \quad \omega_0 = 2\gamma T^{-1}, \tag{6.2}$$

We substitute relations (6.2) into condition (4.14)

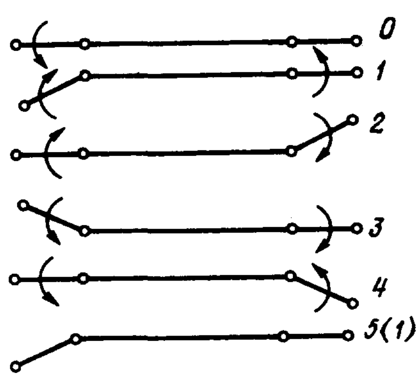

Fig. 4.

$$m_0 l[4\sqrt{2}\gamma(\gamma^2+1)^{1/2}T^{-2}+(4\gamma l T^{-2}+gk)a^{-1}]\le m_1 gk \tag{6.3}$$

To keep the body fixed in the slow phases of the longitudinal motion, it is sufficient for its parameters to satisfy condition (6.3). The average velocity of the longitudinal motion is defined by

$$v_1=\Delta_0 x(2T)^{-1} \tag{6.4}$$

The quantity $\Delta_0 x$ is given by formula (6.1).

7. LATERAL MOTION

The lateral motion of the multilink system is simpler than the longitudinal motion. Again, it is assumed that, at the initial instant of time, the multilink system is at rest and forms a segment parallel to the x axis (see state 0 in Fig. 5). We have $\theta = \alpha_1 = \alpha_2 = 0$.

The lateral motion can be provided by the following combination of elementary motions.

1. Slow motion, during which both end links O_1C_1 and O_1C_2 rotate about the corresponding joints C_1 and C_2 by angles $-\gamma$ and γ respectively. As a result, the multilink system is transferred to state 1 in Fig. 5, in which $\alpha_1 = -\gamma$ and $\alpha_2 = \gamma$.

2. Rapid motion of type 2 from Section 3, as a result of which the angle α_1 changes from $-\gamma$ to γ, and the angle α_2 changes from γ to $-\gamma$. The multilink system is transferred to state 2 in Fig. 5.

3. Slow motion, in the course of which the angle α_1 changes from γ to $-\gamma$, and the angle α_2 changes from $-\gamma$ to γ. The multilink system is transferred to state 3 in Fig. 5.

State 3 is identical to state 1. Further, the cycle of one rapid motion of type 2 and one slow motion can be repeated any number of times. To bring the system back to its initial state at the end of the motion, it is sufficient to execute a slow motion, changing, in the course of the motion, the angle α_1 from $-\gamma$ to 0 and the angle α_2 from γ to 0.

During the lateral motion, as follows from Section 5, the orientation of the body remains unchanged ($\theta \equiv 0$), and the longitudinal displacement is zero ($\Delta x = 0$). The full lateral displacement of the body over a cycle, according to formula (5.11), is equal to

$$\Delta_0 y = 4m_0 m^{-1} l \sin\gamma, \quad m=2(m_0+m_1) \tag{7.1}$$

The duration of the cycle of the lateral motion is approximately equal to the time T of the slow phase. In the notation of (3.1) we have

$$|\Delta\alpha| = 2\gamma, \quad \omega_0 = 4\gamma T^{-1}, \quad \varepsilon_0 = 8\gamma T^{-2} \tag{7.2}$$

Substituting (7.2) into condition (4.16), we obtain

$$8m_0\gamma(4\gamma^2+1)^{1/2}T^{-2}\le m_1 gk \tag{7.3}$$

Condition (7.3) is the sufficient condition for the body to be immobile in the slow phase of the lateral motion. The average velocity of the lateral motion is equal to

$$v_2=\Delta_0 y T^{-1} \tag{7.4}$$

The quantity $\Delta_0 y$ is given by formula (7.1).

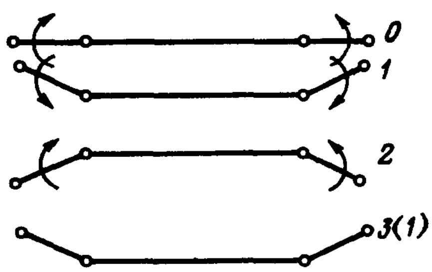

Fig. 5.

8. ROTATION OF THE MULTILINK SYSTEM

In order to turn a multilink system of initial rectilinear form ($\theta = \alpha_1 = \alpha_2 = 0$), we shall perform the following motions ($\alpha_1 = \alpha_2$ in all cases).

1. A slow motion to change the angles $\alpha_1 = \alpha_2$ from 0 to α^0. The multilink system will be transferred from state 0 to state 1 in Fig. 6.
2. Rapid motion of type 3, during which the angles $\alpha_1 = \alpha_2$ change from α^0 to α^1. In this case the body will rotate by an angle of $\Delta\theta$, and the multilink system will be transferred to state 2 in Fig. 6.

These motions can be repeated. To transfer the multilink system from state 2 to a rectilinear state, it is necessary to carry out a slow motion, changing the angles $\alpha_1 = \alpha_2$ from α^1 to 0. The angle of rotation of the multilink system is defined by formula (5.13). The sufficient conditions for the body to be immobile in the slow phase of rotation have the form (6.3), where $\gamma = |\alpha^1 - \alpha^0|$.

9. EXAMPLES

We will specify the following values of the parameters

$$\gamma = \pi/4, \quad k = 0.3, \quad g = 9.8 \text{ m s}^{-2} \tag{9.1}$$

and examine two forms of multilink system: "large" and "small". For the large multilink system, it will be assumed that

$$m_0 = 1 \text{ kg}, \quad m_1 = 1.6 \text{ kg}, \quad a = l = 0.2 \text{ m}, \quad T = 1 \text{ s} \tag{9.2}$$

and for the small multilink system it will be assumed that

$$m_0 = 0.1 \text{ kg}, \quad m_1 = 0.16 \text{ kg}, \quad a = l = 0.02 \text{ m}, \quad T = 0.5 \text{ s} \tag{9.3}$$

Substituting the values given in Eqs (9.1)–(9.3) into conditions (6.3) and (7.3), it can be shown that both of these conditions are satisfied for both multilink systems. The average velocities of longitudinal motion (6.4) and lateral motion (7.4) for the large multilink system (9.2) are as follows:

$$v_1 = 0.02 \text{ m s}^{-1}, \quad v_2 = 0.1 \text{ m s}^{-1}$$

and for the small multilink system (9.3) they are as follows:

$$v_1 = 0.01 \text{ m s}^{-1}, \quad v_2 = 0.05 \text{ m s}^{-1}$$

To maintain the motions, the torques M_i developed by the motors must be significantly greater (by an order of magnitude) than the moments of the friction forces m_0gkl. For the large multilink system (9.1) these moments should be of the order of 4–8 N m, and for the small multilink system (9.2) of the order of 0.1–0.2 N m.

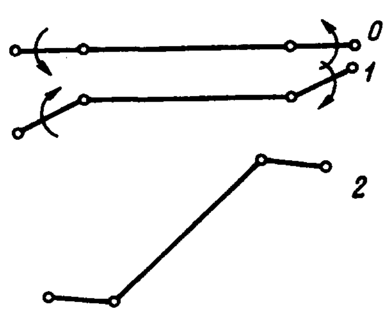

Fig. 6.

10. CONCLUSIONS

It has been shown that a plane multilink system can move along a rough plane in different directions and rotate under the action of internal control torques perpendicular to the plane of motion. Design methods have been proposed for the longitudinal motion, lateral motion and rotation, and sufficient conditions for their feasibility have been found. The displacements and average velocities of motion have been estimated. The sufficient conditions have a large safety margin; these motions are also feasible with less stringent requirements concerning the parameters of the multilink system and the characteristics of the motions.

The proposed forms of motion consist of slow phases, during which the body (the central link) of the multilink system remains stationary, and rapid phases, in which, due to intense "paddling" of the end links, the body moves in the necessary direction. Computer simulation has confirmed the feasibility of the longitudinal and lateral motions of the multilink system with more accurate and detailed allowance for all the factors involved.

The mode of motion considered is distinguished by the fact that the moving body performs plane motion and is in contact with the plane the whole time at the same points. It is this characteristic of the motion that is possessed by snakes and some other limbless animals moving solely by bending their bodies, which lie on the ground the whole time.

Certain important features of the mode of motion considered will be pointed out.

1. Since the motion of the system is planar, the apparatus may be small in height. For comparison, note that the height of wheeled and walking systems has a lower limit, set by the size of the wheels or legs.

2. The apparatus may be equipped with only two motors, whereas for walking device at least two motors per leg are required.

3. Both the design of the device and the control mode are extremely simple.

By virtue of the above, the mode of motion considered may be useful as a possible form of movement for small mobile robots (mini- and microrobots).

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